Numerical Errors in the Computation of Impedances by FDTD Method and Ways to Eliminate Them

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***Abstract*— This paper presents the numerical errors in the computation of impedances by FDTD method. As shown by examples of the paper, substantial numerical errors can be introduced due to the spatial and time offsets of voltages and currents calculated from corresponding field variables in FDTD computations. An impedance calculation formula that completely eliminates such error is also presented.**

I. Introduction

THE CALCULATION of characteristic impedances of transmission structures is often needed in the modeling of microwave integrated circuit components and interconnections in digital circuits and electronic packaging [1], [2]. In typical FDTD computations, the impedance of a transmission structure is obtained by the ratio of a voltage over a current in the frequency domain. Transient voltages are calculated from the line integration of the electric field component between conductors, and currents are from the loop integration of the magnetic field components around conductors. Transient voltages and currents are then Fourier transformed into the frequency domain for the calculation of impedances.

According to Yee's FDTD scheme, the electric and magnetic fields are located at different positions and computed alternatively in time [3]. Therefore, the voltage and the current computed by the integration of electric and magnetic field components are also offset from each other by half a space-step along the direction of transmission structures and half a time-step in time. These spatial and time offsets will introduce some numerical error in the impedance calculated. However, it appears that the effect of this error has not been discussed in past literatures. In this paper, the influence of the error due to the spatial and time offsets of voltages and currents on the impedance calculation is evaluated. It will be demonstrated that a nonnegligible numerical error can be introduced in the impedance calculation, and such an error can actually be eliminated completely by the impedance calculation formula presented in this paper.

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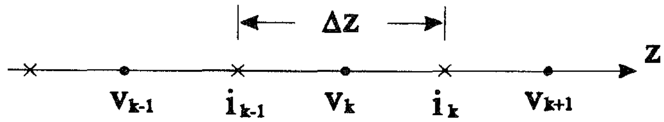

Fig. 1. Locations of voltages and currents.

II. Numerical Examples

The error caused by the spatial and time offsets of voltages and currents in the impedance calculation, and the ways to reduce and eliminate this error, can be easily evaluated through numerical examples. First, let us consider a 50-Ω coaxial line. The reason to choose the coaxial line as an example is that its characteristic impedance is analytically known. That is, for the TEM wave in an ideally lossless coaxial line, the real part of the characteristic impedance is a frequency-independent constant and its imaginary part is exactly zero. To make the characteristic impedance of the coaxial line be 50 Ω, physical dimensions of the coaxial line are chosen as follows. The outer diameter of the inner conductor is 4.137 mm, and the inner diameter of the outer conductor is 9.525 mm. The medium between the inner and the outer conductors is air. The space-steps of the FDTD mesh are: $\Delta r = 0.5388$ mm and $\Delta z = 1.0$ mm. The time step Δt is chosen as 0.5 $\Delta r\sqrt{\epsilon_0\mu_0}$. A pulse propagating along the coaxial line, which is oriented in the z direction, is simulated by the FDTD computation in cylindrical coordinates. Absorbing boundaries are placed at two ends of the coaxial line [4]. Here, the voltage is the line integration of the electric field component E_r from the inner conductor to the outer conductor in the radial direction, and the current is the loop integration of the magnetic field component H_φ around the inner conductor.

If the characteristic impedance of the coaxial line is simply calculated as the ratio of the voltage and the current as follows:

$$Z_1(\omega) = \frac{V_k(\omega)}{I_k(\omega)} \tag{1}$$

where the locations of the voltage V_k and the current I_k are offset by half a space-step as shown in Fig. 1, the real and imaginary parts of the characteristic impedance Z_1 are plotted as curve 1's in Fig. 2(a) and (b), respectively. As can be seen, although the DC values of the characteristic impedance are correct, numerical errors appear at frequencies other than DC

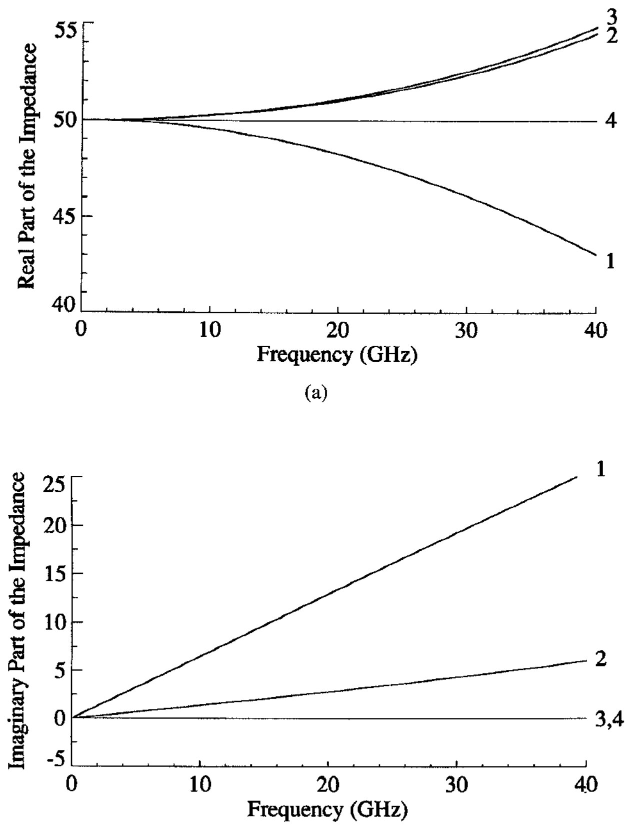

Fig. 2. Characteristic impedance of a 50-Ω coaxial line from FDTD computations. Curves 1–4 correspond to Z_1 to Z_2, expressed in (1), (2), (3), and (5), respectively. (a) Real part. (b) Imaginary part.

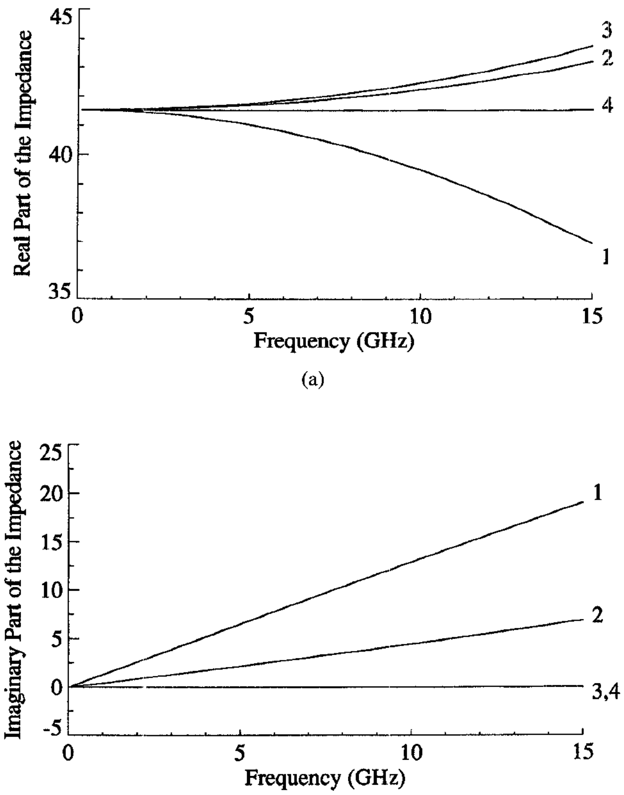

Fig. 3. Characteristic impedances of a strip line from FDTD computations. Curves 1–4 correspond to Z_1 to Z_2, expressed in (1), (2), (3), and (5), respectively. (a) Real part. (b) Imaginary part.

and increase with the increase of the frequency. Numerical solutions at high frequencies can be far off from analytical ones.

As shown in Fig. 1, voltages and currents are located by half a space-step away. One way to reduce the error due to this space difference is to take the average value of the currents on both sides of the voltage V_k, and the characteristic impedance is calculated as

$$Z_2(\omega) = \frac{2V_k(\omega)}{I_k(\omega) + I_{k-1}(\omega)}. \tag{2}$$

The characteristic impedance calculated by (2) are plotted as curve 2's in Fig. 2(a) and (b). As we can see, they are much closer to analytic solutions than those of Z_1. But errors in Z_2 are still apparent.

The time difference between the voltage and the current can be compensated by multiplying V_k by a factor $e^{-j\omega\Delta t/2}$as follows:

$$Z_3(\omega) = \frac{2V_k(\omega)e^{-j\omega\Delta t/2}}{I_k(\omega) + I_{k-1}(\omega)}. \tag{3}$$

Results from (3) are shown as curve 3's in Fig. 2(a) and (b). Although the real part of Z_3 is not improved over that of Z_2, the imaginary part of Z_3 is reduced to zero, which is the expected correct value.

The numerical error that exists in the results from (3) is due to the following reason. In order to calculate the impedance, the current needs to be at the same position as the voltage V_k. Let us call the current at the same position as the voltage V_k as $I_{k-1/2}$. In (2) and (3), the average of I_k and I_{k-1} is an approximation of $I_{k-1/2}$, but not exactly equal to $I_{k-1/2}$. Therefore, the numerical error due to the spatial difference of the voltage and the current in (2) and (3) is reduced from that in (1), but is not totally eliminated. As a matter of fact, $I_{k-1/2}$ is related to I_k and I_{k-1} by the following expression:

$$I_{k-1/2}(\omega) = \sqrt{I_{k-1}(\omega)I_k(\omega)} \tag{4}$$

and the characteristic impedance calculated by

$$Z_4(\omega) = \frac{V_k(\omega)e^{-j\omega\Delta t/2}}{\sqrt{I_{k-1}(\omega)I_k(\omega)}} \tag{5}$$

are plotted as curve 4's in Fig. 2(a) and (b). As can be seen from Fig. 2(a) and (b), both the real and imaginary parts of the characteristic impedance computed by (5) coincide with analytical solutions.

The second example presented below is a strip line. A metal strip of zero thickness and 2 mm in width is placed at the center between two metal planes separated by 4 mm. The medium between two metal planes has a relative dielectric constant $\epsilon_r = 4$. A pulse propagating along the strip line, placed along the z direction, is simulated by the FDTD computation in rectangular coordinates. The computation parameters are chosen as: $\Delta x = \Delta y = \Delta z = 1$ mm, $\Delta t = 0.5\Delta x\sqrt{\epsilon_0 \epsilon_r \mu_0}$. The characteristic impedances calculated by Z_1, Z_2, Z_3 and Z_4, as defined in (1), (2), (3), and (5) respectively, are shown in Fig. 3(a) and (b) as curves 1–4. By observing results in Fig. 3, the same conclusion can be obtained from the case of the strip line as from that of the coaxial line. That is, the characteristic impedance calculated from (5) completely eliminates the numerical error caused by the spatial and time differences of the voltage and current.

III. Conclusion

Examples of computations of the characteristic impedance of transmission structures show that substantial numerical errors can be generated due to the spatial and time difference between voltages and currents by FDTD method. Such numerical errors can be eliminated by the impedance calculation formula presented in this letter.

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